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ARTICLES

Fluctuations in self-organizing systems

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Self-organized criticality (SOC) in a wide variety of systems is seen to arise as a consequence of a singularity in the diffusion coefficient of the hydrodynamic limit. We demonstrate that this description is valid for several models on a closed system and observe that it can break down if the driving is sufficiently strong on the open systems where SOC is observed. In this case fluctuations play an important role, and if fluctuations are large enough then pure power laws in event-size distributions are observed. In contrast, when diffusion holds on SOC systems the characteristic event size diverges sublinearly in the system size. We derive an exponent inequality which provides a necessary condition for the singular-diffusion description to hold on the open driven system. The inequality involves the order of the diffusion singularity, the driving rate, and standard critical exponents.

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I. INTRODUCTION

In this paper we examine the role of fluctuations in the behavior of self-organizing systems. In particular, we consider cellular automata which exhibit the behavior referred to as "self-organized criticality (SOC)"—nontrivial scaling of certain event-size distributions as a function of system size without the explicit tuning of a parameter—which was first introduced by Bak, Tang, and Wiesenfeld (BTW) [1]. The prototypical example of this phenomenon is a "sandpile" model, in which "sand" is added one grain at a time to randomly selected sites on a d-dimensional integer lattice, and when the local height or slope exceeds a threshold, sand falls according to a prescribed set of rules. Since this concept was first introduced, simulations indicate that a wide variety of models have qualitatively similar behavior [2–5].

One key to understanding the scaling behavior of many of these systems is the observation that on the closed system the hydrodynamic limits are singular-diffusion equations describing the evolution of a conserved quantity [6,7]. That is, in an appropriately rescaled limit, the hydrodynamic description of the evolution of the conserved density (e.g., height or slope), which we will call ρ, is given by a diffusion equation of the form

$$\frac{\partial \rho}{\partial t}=\nabla\cdot[D(\rho)\nabla\rho] \ , \tag{1}$$

where the diffusion coefficient not only depends on the local density, but in fact has a singularity at a critical value ρ_c,

$$D(\rho)\sim\frac{1}{(\rho_c-\rho)^{\phi}} \ . \tag{2}$$

Simply stated, a diverging diffusion coefficient is an indication that the typical transition lengths (event sizes) diverge as the density approaches the critical value ρ_c. In some cases this hydrodynamic limit has been established rigorously [7] or analytically [8], and in other cases the singularity in the diffusion coefficient has been ascertained numerically via relaxation times for density perturbations or variances of tagged particles [9,10].

In this paper we present an exponent inequality that provides a condition under which the hydrodynamic description is expected to break down on the open driven system due to fluctuations in the empirical density. As we shall see, the key ingredients are the order of the singularity ϕ and standard critical exponents. In Table I we present a summary of the diffusion singularities as well as other exponents, which will be important in this

TABLE I. Critical behavior of SOC systems. Here ρ_c is the critical density, ϕ is the order of the diffusion singularity (2), ν is the exponent characterizing the divergence of the spin-spin correlation length ξ, η is the exponent characterizing the small k behavior of the structure function (7), $\hat{\nu}$ is the exponent describing the divergence of the characteristic event size $\hat{\xi}$ as in (19) ($\hat{\nu}>0$ for any SOC system), and μ is the exponent describing the scaling of density fluctuations on an individual site as in (12). For the BTW model we have taken the threshold value $h_c=4$. For the limited local and limited nonlocal models we have taken $z_c=n=2$. For the unlimited local model we have taken $z_c=4$ and $n=2$. Apart from changes that lead to trivial behavior, different values of the parameters should only alter the value of the critical density, ρ_c, leaving the exponents unchanged. When blanks appear in place of exponents, the values have not been measured, or, in the case of the limited local model, our measurements have not led to well-defined values.

Model	ρ_c	ϕ	$\nu(-2+\eta)$	$\hat{\nu}$	μ
Two-state	1	3	0	1	1
BTW	2.12 ± 0.005	2.3 ± 0.1	0	0.74 ± 0.01	0
Count	$\frac{1}{2}$	2.25 ± 0.15	0.95 ± 0.15	1 ± 0.1	0
Limited local	$\frac{3}{2}$	4			0
Limited nonlocal	$\frac{3}{2}$	4			0
Unlimited local	2	2			0

paper, for the collection of models that we have considered to date. The list is by no means exhaustive, but rather illustrates the range of behaviors that can be observed in these simple models.

The diffusion description above is derived for the closed system. That is, the diffusion limit describes the relaxation of a nonequilibrium density profile, which is bounded away from the singularity ρ_c on a closed system such as a torus, when the number of sites diverges. Presumably this description also holds in the thermodynamic limit, i.e., on the full integer lattice. For SOC the utility of this result comes from *assuming* that we can use this description to determine the stationary states of the open driven systems. In particular, one solves the stationary equation

$$\frac{\partial \rho}{\partial t}=\nabla\cdot[D(\rho)\nabla\rho]=0\ , \tag{3}$$

with boundary conditions appropriate to the driving mechanism [6]. The resulting solution yields a steady-state profile on an N^d-site system which scales as $\rho_c-\rho_N(x)\sim N^{-b}$ for all values of $x\in(0,1)^d$. The exponent b is related to ϕ and to the boundary conditions. Furthermore, if one knows enough about the equilibria at local density ρ and about the (microscopic) transition rates, then the scaling of event-size distributions can be calculated from the scaling of $\rho_N(x)$.

However, for the open driven systems encountered in SOC, the validity of the hydrodynamic limit is not automatic. If one assumes that the diffusion limit holds, then the density converges to the singularity as the system size N diverges, raising the possibility of an illegal interchange of limits. More explicitly, there are two primary ways in which the hydrodynamic description can break down: (1) Fluctuations in the empirical density can force the local density over the singularity in macroscopic regions of the system. (2) Local equilibrium can break down, since the open systems sustain a flux which, under rescaling, is typically diverging with system size.

The main results in this paper are as follows.

(a) The derivation of conditions under which a breakdown of hydrodynamics occurs due to fluctuations, assuming the validity of local equilibrium through the crossover. This is the topic of Sec. II.

(b) A demonstration of how this sort of breakdown is consistent with our numerical observations of several models, including the BTW sandpile model. Our simulations indicate that when fluctuations become relevant, pure power laws in event-size distributions are seen in one dimension, and in any dimension under sufficiently hard driving. On the other hand, when diffusion holds the characteristic event size diverges sublinearly with the system size. This is discussed in Sec. III.

(c) Results from simulations of the limited local sandpile model which indicate that the breakdown of local equilibrium occurs before fluctuations begin to dominate. This is also presented in Sec. III. General conditions for the failure of local equilibrium are a topic of current research.

(d) A summary of values of the order of the diffusion pole as well as other statistical-mechanical exponents for a variety of SOC systems (Table I). Some of these results have not been published previously, and, with a few expections, are determined numerically.

Before proceeding to the next section we briefly discuss certain aspects of Table I. One interesting feature is that the limited local and limited nonlocal sandpile models, which differ only slightly in their toppling rules, both have the same diffusion singularity $\phi=4$, while in Ref. [2] it was reported that the multifractal scaling exponents for these two models are different. These results are not inconsistent because the diffusion singularity represents the long-time, large-wavelength behavior of these systems, while the scaling laws refer to the rates of microscopic events in the systems. It is also interesting to observe that the unlimited local model has a different value of ϕ. The behavior of this model is very different from the other systems we have considered, due to the fact that on the open system using the microscopic transition rules it is easily shown that states with density below ρ_c are transient (i.e., when the density of the open system exceeds ρ_c it can never drop below ρ_c). In this case we find that the average density is always greater than ρ_c for this relatively simple reason, and as the system size increases the density monotonically approaches the diffusion singularity from above. For sufficiently hard driving the BTW sandpile model also approaches ρ_c from above. However, in this case there are no such constraints on the density, and the behavior is only observed after the diffusion description has clearly failed.

II. FLUCTUATIONS AND THE FAILURE OF HYDRODYNAMICS

We now derive an exponent inequality that indicates when fluctuations destroy the diffusion hydrodynamics on the open drive system. Before we begin it is important to clarify exactly what the hydrodynamic limit is describing. The N^d-site system is rescaled into $[0,1]^d$. We denote the spin (height, slope) at site i/N by $\zeta(i/N)$. The object of attention is the empirical density $\hat{\rho}_{l,N}(x)$, which is the empirical (random) density of the conserved quantity in a box of size $2l$ which is a subset of $[0,1]^d$.

$$\hat{\rho}_{l,N}(x)=(2lN)^{-d}\sum_{|y-x|(\leq lN)}\zeta(y/N)\ , \tag{4}$$

where $|y-x|$ is the maximum difference of coordinates, so that the sum above is over a box of size $(2lN)^d$ sites. One should think of l as being very small (one takes $N\rightarrow\infty$, and then $l\rightarrow 0$), and we will omit it in subsequent notation. The hydrodynamic description is valid when the solution to the diffusion equation $\rho_N(x)$ adequately approximates the empirical density $\hat{\rho}_N(x)$. Since the solution ρ_N and expected density $E[\hat{\rho}_N]$ are both converging to the singularity ρ_c, one must clarify what is meant by the phrase "adequately approximates."

The appropriate criterion is that as $N\rightarrow\infty$ the fluctuations in the empirical density are governed by a linearized [about $\rho_N(x)$] diffusion equation with an appropriate stochastic driving. That is, we linearize (1) about the steady-state solution $\rho_N(x)$ of (3) which is simultaneously converging to the singularity: $\rho_c-\rho_N(x)\sim N^{-b}$. Note that this approach differs significantly from that of Refs. [11,12], which assume that the behavior of the system is smooth in the continuum limit, i.e., that there are no singularities in the diffusion coefficient or the conductivity. In our case the singularities are explicitly taken into account, and are, in fact, a crucial ingredient.

Denoting the stochastic fluctuations about ρ_N by $s_N\equiv(\mathrm{Var}\,[\hat{\rho}_N])^{1/2}\sim N^{-a}$ and the gap $\rho_c-\rho_N(x)\sim N^{-b}$, hydrodynamics fails in the above sense if $b\geq a$; that is, if the fluctuations decay more slowly than does the gap from the singularity, so that the empirical density fluctuates above the singularity. See Fig. 1 for an illustration of this behavior in the BTW sandpile model. To see that this is the correct criterion, we linearize the singular diffusion equation (1) and add an appropriate noise term [13] (we include this term solely for completeness). We will, however, keep terms to second order in the gradient expansion:

$$\frac{\partial s_N}{\partial t}=\nabla^2[D(\rho_N)s_N]+\nabla\cdot\left[D'(\rho_N)s_N\frac{\partial s_N}{\partial x}\right]+\nabla\cdot\{[\sigma(\rho_N)+(\sigma^{-1/2})\sigma'(\rho_N)s_N]j_t\}\ , \tag{5}$$

where j_t is the standard Gaussian noise. The full equation and boundary conditions are required to ascertain the effect of the driving mechanism on local equilibrium. However, here we are only interested in the effect of the fluctuations on the size of the higher-order terms. Rescaling spatial variables by N^{-1} and noting that $D(\rho_N)\sim N^{b\phi}$, we see that the first term on the right-hand side scales like $N^{b\phi-2-a}$ while the second term scales like $N^{b(\phi+1)-2-2a}$ (when the second term is relevant, all higher-order terms grow at least this quickly), implying that the higher-order terms cannot be neglected when $b\geq a$. This breakdown can also be seen in the third term involving the conductivity σ [14].

For the breakdown condition $b\geq a$ to be of the greatest use, we need to determine a and b from more readily available information about the underlying system. To this end, we will now use scaling arguments to calculate a and b from the boundary conditions and the statistical-mechanical properties of the system.

A. Scaling of the mean density

The value of b can be determined self-consistently, by balancing the time scales associated with driving and diffusion. The idea here is that density changes which develop on the time scale of the driving mechanism must relax at the rate associated with the diffusive mechanism. Therefore these rates must balance as long as the diffusion description is valid. Referring to (1), we rescale x by N^{-1} and, since $\rho_c-\rho_N(x)\sim N^{-b}$, we find that the right-hand side scales like $N^{b\phi-2}$, which implies that the diffusion time must scale as $N^{2-b\phi}$.

Next we focus on the driving time scale. The mass of any injected or removed particle has weight $1/N^dN^b=N^{b-d}$, where the first term is the usual scaling and the correction N^b is due to the fact that we are looking at a gap from the singularity of size N^{-b}. The boundary driving dimension is denoted by d_B, and quantifies the scaling of the driving rate with system size, so that the relevant driving rate scales as N^{b+d_B-d}. Therefore the relevant time scale for the driving mechanism is N^{d-b-d_B}.

Making the topological assumption that a positive fraction of the driving region does not overlap the open boundary (even in the continuum limit) [15], we equate the diffusion and driving times scale to obtain

$$b=\frac{2-d+d_B}{\phi-1}\ . \tag{6}$$

The exponent d_B warrents further elaboration. In the most familiar cases, d_B corresponds to the dimension of the boundary along which the conserved quantity is added to the system. However, d_B need not equal the scaling dimension of the set of points on which the driving mechanism acts. For example, in one-dimensional systems such as the limited local sandpile model or the two-state model, particles can be added at a single boundary point (dimension zero); however, the addition rate can be a function of the system size so that d_B is nonzero. In this way typically well-behaved systems can be forced into regimes in which hydrodynamics is no longer valid. We will elaborate on this point below.

B. Scaling of density fluctuations

The appearance of a singularity in $D(\rho)$ does not imply anything about the critical behavior of the underlying spin system. The singularity is present as a consequence of transition lengths $\hat{\xi}$ diverging (think of site percolation, where spins are uncorrelated, since they are assigned independently, but where there is a phase transition in the cluster size). Therefore, if the spin-spin correlation length ξ is uniformly bounded for all densities $\rho \le \rho_c$, fluctuations will satisfy the central limit theorem, i.e., $a = d/2$. This turns out to be the case for the BTW sandpile model (see Table I).

There is, however, the possibility that the spin system is critical at ρ_c, as happens, for example, in the limited local model. The size of the density fluctuations is related to the small-k behavior of the structure function $\hat{S}(k)=\sum_x e^{ik\cdot x}S(x)$, where $S(x)=\langle[\zeta(0/N) - \langle\zeta(0/N)\rangle][\zeta(x/N)-\langle\zeta(x/N)\rangle]\rangle$ is the correlation function, and x denotes a d-dimensional integer lattice site. The scaling hypothesis for $\hat{S}$ at a critical point is

$$\hat{S}_c(k) \sim |k|^{-2+\eta} , \tag{7}$$

from which one finds that $s_N \sim N^{-(1/2)(d-2+\eta)}$. It should be pointed out that central limit fluctuations occur where $\nu=0$ and, consequently, $\eta=2$. When $\eta<2$ fluctuations are enhanced (e.g., ferromagneticlike correlations) and when $\eta>2$ fluctuations are suppressed (e.g., antiferromagneticlike correlations). In the latter case, some staggered order parameter will have enhanced fluctuations; but here we are only concerned with the fluctuations of the density, which can be suppressed at a critical point.

In our problem, however, the system is not *at* the critical point; rather it is converging to it as $N \to \infty$. Denoting the spin-spin correlation length by $\xi(\rho)$ as above, the scaling hypothesis becomes

$$\hat{S}(k,\xi) \sim \begin{cases} |k|^{-2+\eta} & \text{if } k\xi \to \infty \\ \xi^{2-\eta} & \text{if } k\xi \to 0 . \end{cases} \tag{8}$$

Since $\rho_c - \rho_N \sim N^{-b}$, we have $\xi \sim |\rho_c-\rho|^{-\nu} \sim N^{b\nu}$. The variance of the density fluctuations is found to scale as $s_N \sim N^{-a}$, with

$$a = \begin{cases} \frac{1}{2}[d-2+\eta] & \text{when } b\nu > 1 \\ \frac{1}{2}[d+b\nu(-2+\eta)] & \text{when } b\nu \le 1 . \end{cases} \tag{9}$$

We consider the second case only (since ξ is effectively bounded by system size), and conclude that fluctuations are relevant when

$$\tfrac{1}{2}[d+b\nu(-2+\eta)] \le b . \tag{10}$$

C. Exponent inequalities

Assembling the previous results provides a condition under which fluctuations result in the failure of diffusion hydrodynamics. We have assumed the following.

(1) The underlying spin system has a correlation length which scales like $\xi \sim |\rho_c-\rho|^{-\nu}$ and a structure function which scale like $\hat{S}(k,\xi) \sim \xi^{2-\eta}$.

(2) The singular-diffusion coefficient scales like $D(\rho) \sim |\rho_c - \rho|^{-\phi}$ and the driving rate δ scales like $\delta(N) \sim N^{d_B}$.

Substituting (6) into (10), we find that the condition on the driving rate which implies failure of the hydrodynamic description due to fluctuations is

$$d_B \ge d\left[\frac{\phi-1}{2-\nu(-2+\eta)}+1\right]-2 . \tag{11}$$

Finally, it should be pointed out that, for example, in the two-state models introduced and studied in Refs. [6,7], the situation can arise where the correlation length remains infinite, but where the single-site variance scales nontrivially. This occurs because the diffusion singularity coincides with the upper bound on allowed densities. Consequently, fluctuations are suppressed simply because fluctuations above ρ_c cannot occur. In this case, define the scaling of the single site variance $\tilde{s}_N$ to be

$$\tilde{s}_N^2 \equiv \operatorname{var}_\rho[\zeta(i)] \sim |\rho_c-\rho|^{-\mu} \sim N^{-b\mu} , \tag{12}$$

with b given in (6). Defining a modified correlation function

$$S(x)=\tilde{s}_N^{-2}\left\langle\left[\zeta\left[\frac{0}{N}\right]-\left\langle\zeta\left[\frac{0}{N}\right]\right\rangle\right]\times\left[\zeta\left[\frac{x}{N}\right]-\left\langle\zeta\left[\frac{x}{N}\right]\right\rangle\right]\right\rangle , \tag{13}$$

the extension of (9) to this setting is

$$s_N \sim \tilde{s}_N N^{-(1/2)[d+b\nu(-2+\eta)]} = N^{-(1/2)[d+b\nu(-2+\eta)-b\mu]} \tag{14}$$

and the exponent inequalities such as (10) are easily modified:

$$\tfrac{1}{2}[d+b\nu(-2+\eta)-b\mu] \le b . \tag{15}$$

III. OPEN-SYSTEM RESULTS

In this section we discuss results from numerical simulations of the first four models listed in Table I, with emphasis on the role of fluctuations and the validity of local equilibrium.

A. Two-state models

We begin with the one-dimensional two-state model of Ref. [6]. This is one of a general class of models considered in Ref. [7]. In this system each site is occupied by a 1 or a 0 and the 1's jump at rate 1 to the nearest vacant site to the right, and likewise to the left. For this model we take as the closed system a lattice of length N with periodic boundary conditions. The open (driven) version of the system that we consider has an open boundary at the right edge where 1's may leave the system, and a closed edge on the left where 1's are injected at rate N^{d_B}. The case $d_B=0$, which corresponds to simply injecting particles at rate unity for all system sizes N, was considered in Ref. [6]. However, to see the crossover to failure of diffusion hydrodynamics we will allow d_B to vary, thus introducing a system-size-dependent driving rate.

In this model the equilibria are product measures—that is, sites are uncorrelated. In Ref. [6] it was shown that the diffusion coefficient is $D(\rho)=\sum_{k=0}^{\infty}k^2\rho^{k-1}=(1+\rho)/(1-\rho)^3$ so that $\rho_c=1$ and $\phi=3$. In this case, fluctuations are suppressed because the critical density $\rho_c=1$ is an upper bound on the density. The correlation functions are identically zero, so that $\nu=0$ and $\eta=2$. The single site variance $\bar{s}_N^2=\rho(1-\rho)$ so that $\mu=1$ in (15). This inequality implies that hydrodynamics fails when $b\geq 1$, a situation that corresponds to a finite number of zeros in the system for any system size which is an inherently a noisy limit. To understand this last statement, one must recall that we are interested in the gap between $\rho_c=1$ and the empirical density. Since $1-E[\zeta(i/N)]\sim N^{-b}$, we map each spin according to $\zeta(1/n)\rightarrow N^b[1-\zeta(i/N)]$, where the prefactor N^b expands the gap so that in the limit $N\rightarrow\infty$ the result is a nondegenerate random variable. Assigning mass N^{-1} to each site, one sees that the density field is $\sum N^{b-1}[1-\zeta(i/N)]$. When $b=1$ this is just counting the number of 0's in the range of the sum, which remains a random measure in the limit of large system size.

B. BTW sandpile model

Next we discuss the two-dimensional BTW sandpile model, first introduced and studied in Ref. [1]. In this model sand is added one grain at a time to sites on an $N\times N$ integer lattice, and when the local height equals or exceeds a threshold value $h_c=4$ one grain of sand is transferred to each of the four neighboring sites. To define the closed system we impose periodic boundary conditions, and instead of adding new sand to the system we exchange single grains of sand between nearest-neighbor sites in a randomly selected direction at a fixed rate. This has the effect of conserving sand.

Our numerical results for the critical behavior of this model are obtained on a 128×128 system and are summarized in Table I. The critical height density $\rho_c=2.12\pm 0.005$ and diffusion singularity $\phi=2.3\pm 0.1$ are determined on the closed system via measurements of the rate at which the primary (i.e., sinusoidal) mode of a small-amplitude perturbation relaxes to equilibrium as a function of the systemwide average height as in Ref. [6]. An equilibrium measurement of the distribution of relative displacements of tagged particles as a function of the average height confirms that the variance of the distribution diverges at the same value of the density ρ_c, with a singularity of order $\phi-1=1.3\pm 0.1$. This is consistent with the two-state model for which it was proven [9] that the singularity in the variance of a tagged particle is of order $\phi-1$. Direct measurements of the height-density fluctuations in a box of fixed size indicate that they are independent of the average density; that is, $\nu(-2+\eta)=0$. By varying the system size, we confirm that the decay rate of the fluctuations satisfies the central limit theorem. We also measure the height-height correlation function and find that the correlations decay rapidly (within a few sites), and do not appear to depend on the height density in the system (i.e., $\nu=0$).

On the open system, the conventional driving mechanism involves adding sand to randomly chosen sites across the entire lattice. This corresponds to $d_B=d$, i.e., the dimension of the driving mechanism is equal to the dimension of the system. Substituting the exponents from Table I into (11), we find that fluctuations are relevant when $d_B\geq 1.3$, and we conclude that hydrodynamics should fail for the conventional driving ($d_B=2$). This is a consequence of the very hard driving, which scales like the volume, and the relatively low-order diffusion singularity.

By modifying the driving mechanism we can bring the system back into a regime where the diffusion description holds. The procedure used to drive the system like N^{d_B} is as follows. On each "addition" step to the automata we either add a new particle with probability N^{d_B}/N^d or transfer particles between nearest-neighbor sites with probability $1-(N^{d_B}/N^d)$. Avalanches relax as before. Note that both on the closed system and on the open system when $d_B\neq 2$ the Abelian nature [3] of the model is lost. We adopt the above procedure to decouple the relaxation and driving mechanisms—that is, to allow the occurrence of avalanches without the addition of sand.

The inequality (11) which predicts a crossover at $d_B=1.3$ is consistent with our simulations. However, it is very hard to get into the asymptotic regime, and our best numerical estimates allow us to conclude that fluctuations begin to dominate at some value of d_B in the interval $1\leq d_B\leq 1.5$. To obtain this estimate we consider $d_B=0.5$, 1, 1.5, and 2 for system sizes $N\times N$ with $N=8,16,32,64,128$. In each case we monitor the time-averaged distribution of densities $P_N(\rho)$ both on the middle $\frac{1}{4}$ of the system and on the system as a whole. Examples of situations in which the system does and does not fluctuate above the singularity are illustrated in Fig. 1. To check for convergence, we compare the mean and the width of the Gaussian distributions on disjoint, exponentially growing time intervals until all of the values (for both the middle $\frac{1}{4}$ and systemwide distributions) for consecutive intervals have changed by less than 1% for $N=8,16,32$ or 5% for $N=64,128$. Typically the average densities converge faster than the fluctuations, but in each case the equilibration time is surprisingly long. For example, with $d_B=1$ we accumulate data for over 65 000 events per site, or over five million events on the smallest 8×8 system. For larger values of d_B somewhat fewer events are necessary for convergence, but the events were larger on average, so the equilibration time was comparable. These constraints on equilibration time prevent us from simulating larger systems.

For the case $d_B = 2$ (i.e., the conventional driving), the density fluctuates above the singularity for all system sizes, that is, $s_N > \rho_c - \rho_N$. In fact, for the larger system sizes the average density in the middle $\frac{1}{4}$ of the system is above the singularity, indicative of a clear breakdown of diffusion hydrodynamics. In contrast, for the other values of d_B we have $s_N < \rho_c - \rho_N$, for each of the system sizes we have considered, while according to (11) a breakdown is predicted for large enough system sizes when $d_B = 1.5$. Thus the crucial question is how are the densities and fluctuations scaling with increased system size.

In Fig. 2 we plot the ratio $s_N/(\rho_c - \rho_N)$ as a function of N for $d_B = 0.5, 1, 1.5$. If hydrodynamics holds, i.e., $a > b$, then $s_N \rightarrow 0$ faster than $(\rho_c - \rho_N) \rightarrow 0$, and this ratio should approach zero as N increases. Alternately, when hydrodynamics breaks down and $a < b$ the ratio should diverge. Figure 2 shows the indications of a crossover between $d_B = 1$ and $d_B = 1.5$. In the case of $d_B = 1$, the ratio is scaling to zero roughly as a power of N, while for $d_B = 1.5$ the ratio initially decreases, but with appreciable curvature in the log-log plot, suggesting an eventual divergence.

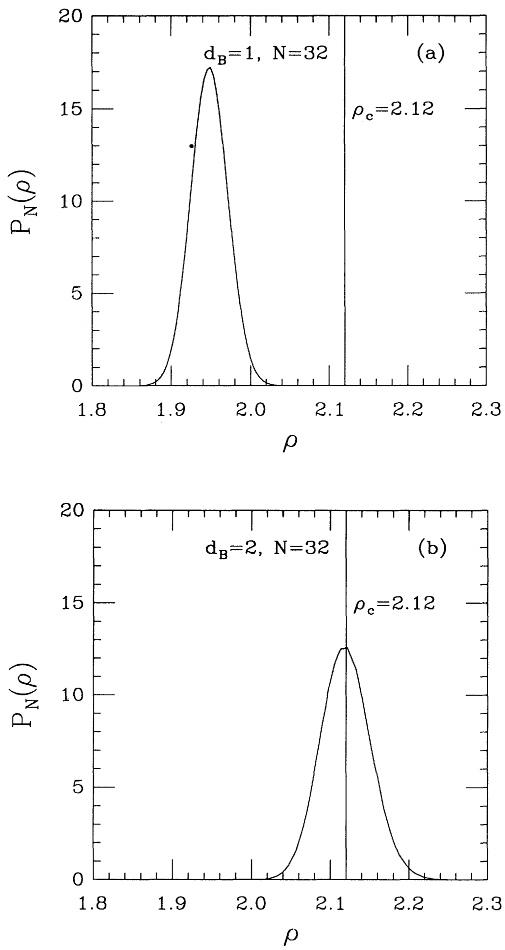

FIG. 1. Distribution of the height density in the middle $\frac{1}{4}$ of the system for the BTW sandpile model. (a) illustrates our results when $d_B = 1$ and $N = 32$, and (b) illustrates our results when $d_B = 2$ and $N = 32$. The data is accumulated until both the mean ρ_N and the width s_N of the distributions and the corresponding distributions for the systemwide density have met the convergence criterion stated in the text. The vertical line at $\rho = 2.12$ represents the diffusion singularity. In (a) $s_N < \rho_c - \rho_N$ and diffusion hydrodynamics holds, while in (b) $s_N > \rho_c - \rho_N$ and diffusion breaks down.

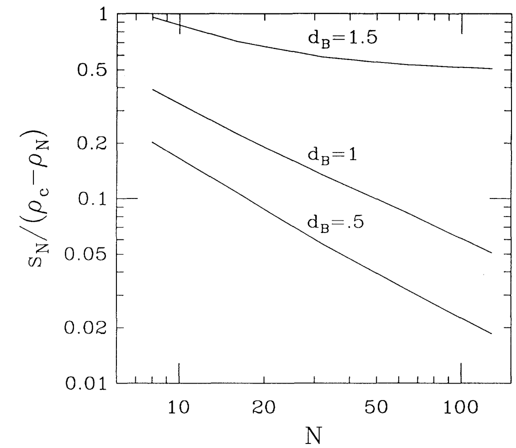

FIG. 2. The ratio of the density fluctuations s_N to the distance from the singularity $\rho_c - \rho_N$ as a function of N. Here ρ_N corresponds to the mean, and s_N to the width, of distributions such as those illustrated in Fig. 1. For $d_B = 0.5$ and $d_B = 1$ the ratio approaches zero as $N \rightarrow \infty$, indicating that diffusion hydrodynamics holds on the open system. However, for $d_B = 1.5$ the ratio curves back up for larger N, indicative of a change of behavior for $1 < d_B < 1.5$. When diffusion breaks down the ratio will diverge as $N \rightarrow \infty$, as we expect to occur for $d_B = 1.5$. The results for $d_B = 2$, where diffusion clearly breaks down, do not appear since in that case not only the fluctuations s_N, but also the mean density ρ_N, exceed the diffusion singularity for the larger systems ($N = 64, 128$).

As previously mentioned, numerical constraints on system size restrict our ability to fit the asymptotic forms in (6) and (9). For example, one can use the average density in the middle $\frac{1}{4}$ of the system to attempt to fit a power law of the form

$$\rho_c - \rho_N = A/N^b , \qquad (16)$$

selecting A and b to minimize the error $\mathcal{E} = \sum[\ln(\rho_c - \rho_N) - \ln(A/N^b)]^2$. The logarithms are taken to avoid unduly diminishing the weighting of the large systems. This one-exponent fit does not work well, yielding values of b that are roughly 10% off the predicted values. The results depart even more from the predicted values when the density of the entire system is used, because of considerable edge effects associated with boundary layers such as those described in Ref. [6]. A similar fit for the fluctuations yields values for a that deviate from the anticipated central-limit value of $d/2 = 1$ by roughly 20%. These results are not adequate to locate to a change in behavior due to fluctuations.

In order to improve our estimates, we assume a correction to scaling of the form

$$\rho_c - \rho_N = \frac{A}{N^b}\left[1 + \frac{D}{N^c}\right] . \tag{17}$$

The curvature in the $d_B = 1.5$ plot of Fig. 2 suggests that such a two-exponent fit is necessary to see the crossover, and in fact it reduces the error, which is obtained by minimizing the function analogous to $\mathcal{E}$ given above with respect to the four parameters A, b, c, and D, by one or more orders of magnitude, and yields much better agreement with the values predicted by (6). The corresponding correction to scaling for the density fluctuations gives similar results. In particular, for $d_B = 1.5$ this fit yields $a = 1.05$ and $b = 1.13$, which agrees well with the calculated values $a = 1.0$ and $b = 1.15$, and confirms our prediction that the density will fluctuate above the singularity for systems that are sufficiently large. These results are illustrated in Fig. 3. By extrapolating the scaling forms we estimate that the value of N above which one would truly see the fluctuations exceed the critical density is $N \sim 10^6$, well beyond the range that is numerically feasible. Finally, a similar fit for $d_B = 0.5$ and $d_B = 1$ gives values consistent with the analytical estimates, and indicates that diffusion holds in these cases.

Lastly, we consider the question of local equilibrium. Note that our results for the BTW model are consistent with the validity of local equilibrium on the open system. This will not be the case for the limited local sandpile model, which shows a clear breakdown of local equilibrium on the open system.

For local equilibrium to be a valid approximation, the correlation functions in any region of the open system must be asymptotically equal to those of the closed system at the same local density in the limit of large system size. In particular, when local equilibrium holds, the exponents listed in Table I, which are measured on the closed system, should describe the behavior of the open system as well, where the average density depends on system size. The results above show that the scaling exponents are consistent with (6) and (9), which implies that the order of the diffusion singularity ϕ and the size of the fluctuations are the same for the open and closed systems.

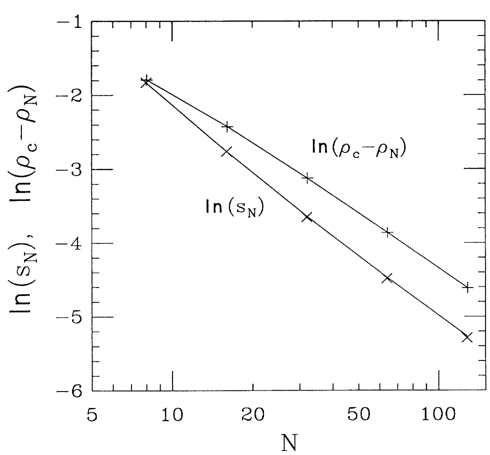

FIG. 3. Results of the two-exponent fits (17) for the density $(\rho_c - \rho_N)$ $(+s)$ and fluctuations $s_N(\times)$ for $d_B = 1.5$ and $N \times N$ systems with $N = 8,16,32,64,128$. (The error bars are much smaller than the sizes of the data-point markers). The best fit agrees well with the data and gives estimates $b = 1.13$ and $a = 1.05$. Note that for the system sizes we consider $s_N < (\rho_c - \rho_N)$; however, the s_N data curves downward with increasing system size, while the $(\rho_c - \rho_N)$ data curves upward. An extrapolation of the fit predicts a dominance of fluctuations, $s_N > (\rho_c - \rho)$, at system sizes in excess of $N = 10^6$.

One can also look at event-size distributions to check the validity of local equilibrium. We denote the equilibrium probability of an event involving n sites by $P(n)$. The singularity in $D(\rho)$ suggests that near ρ_c

$$P(n) \sim \frac{1}{n^{\alpha}} g(n/\hat{\xi}^d) , \tag{18}$$

where $\hat{\xi}$ is the length scale characterizing the (typically exponential) cutoff in the event-size distribution,

$$\hat{\xi} \sim 1/(\rho_c - \rho)^{\hat{\nu}} . \tag{19}$$

Figure 4(a) shows a scaling collapse of $P(n)$ with respect to the distance from the singularity $(\rho_c - \rho)$ for the closed system. The result $\hat{\nu} = 0.74 \pm 0.01$ is obtained from this collapse and agrees with the value obtained previously in Ref. [16]. Figure 4(b) illustrates the corresponding results for the open system with $d_B = 1$, where the average density in the middle $\frac{1}{4}$ of the system is used in the collapse. Some deviation between the exponents which best collapse the closed and open system distributions does arise. However, the difference is sufficiently small that we may attribute it to the curvature of the density profiles on the open system. This explanation does not imply a breakdown of local equilibrium. Instead, when the average density is not constant one should in principle integrate (18) over the spatially varying density to obtain the systemwide events distribution. If we attempt to collapse the open system data [Fig. 4(b)] with the closed system exponents, we observe that the discrepancy between the open and closed systems decreases systematically as system size increases (and consequently the spatial variation decreases due to decreased edge effects) supporting this explanation.

Finally, it is worth noting that with the conventional driving $d_B = 2$ the event-size distribution $P(n)$ is substantially altered. In that case, we cannot possibly collapse the data with $(\rho_c - \rho)$ as in Figs. 4(a) and 4(b) because, as previously mentioned, for the large systems the average density is greater than ρ_c. In this case we can, however, perform a finite-size scaling collapse with N. The result is illustrated in Fig. 4(c). Here the cutoff is given by the system size $\hat{\xi} = N$, and the event-size distribution is essentially a pure power law. In contrast, when diffusion holds on the open driven system [as in, for example, Fig. 4(b)], when we perform a scaling collapse with respect to the system size N rather than the density $\rho_c - \rho$, we observe that the characteristic event size scales like $\hat{\xi} \sim N^{b\hat{\nu}}$, with $b\hat{\nu} < 1$, i.e., the divergence is sublinear in the system size. In that case, under the spatial rescaling of N^{-1} that leads to the diffusion limit, even the largest events correspond to local transitions. In contrast, when $\hat{\xi} \sim N$ typical

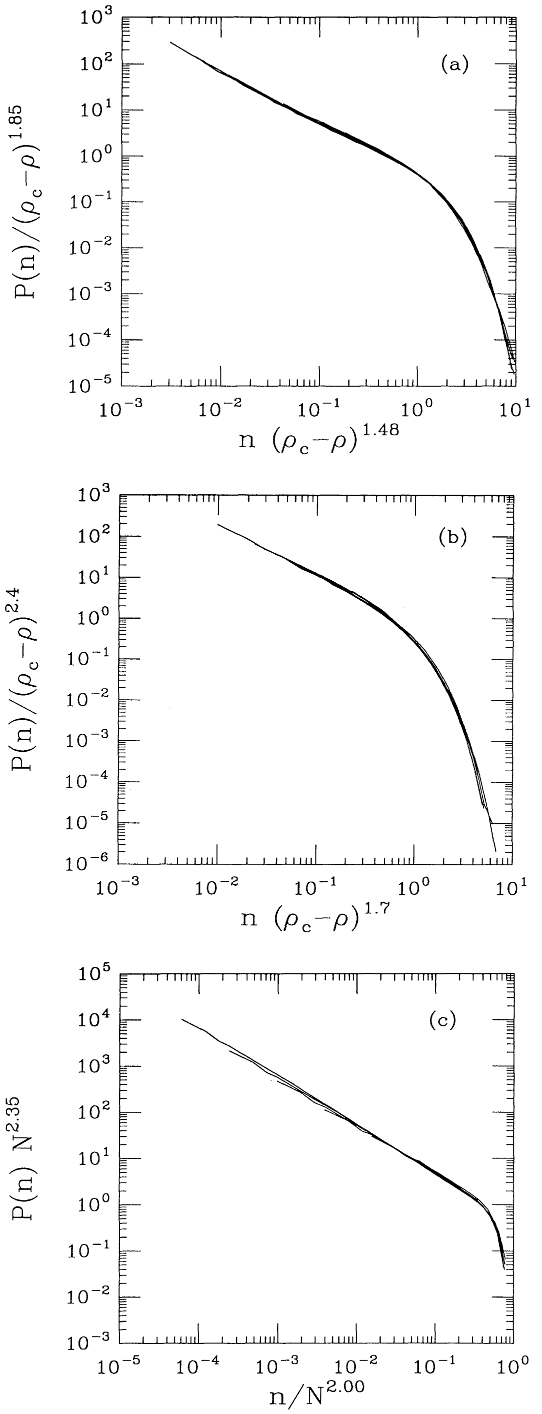

FIG. 4. Scaling collapse of the distribution $P(n)$ of events involving n sites. (a) illustrates our results for the closed system, where the data is rescaled by the distance from the singularity. The x-axis rescaling is $d\hat{\nu}$ from which we extract the event-size correlation length exponent $\hat{\nu}=0.74$. (b) illustrates the corresponding results for the open system with $d_B=1$. In this case we estimate that $\hat{\nu}=0.85$. (c) illustrates our results for $d_B=2$. In this case we rescale by the system size, and the x-axis rescaling is $db\hat{\nu}=2$. Thus the distribution cuts off at events of size $\hat{\xi}\sim N$, and the distribution is essentially a pure power law extending from the smallest events out to the system size.

events are nonlocal, involving a finite fraction of the system. In general, we expect that when the typical event size is of order the system size diffusion hydrodynamics must fail, i.e., local events are a necessary condition for diffusion to hold.

C. Count model

We introduce and briefly discuss another automaton which exhibits both the failure of hydrodynamics under hard driving as well as nontrivial critical behavior at the diffusion singularity. In this system, as in the two-state model, each site is either a 1 or a 0. The rules are as follows. If a 1 is located at site i, then at rate unity the 1 moves to the first site $k>i$ such that there are more 0's than 1's in the intermediate interval. Specifically,

$$k=\inf\left\{j>i:\ \sum_{m=i+1}^{j}\zeta(m)<\sum_{m=i+1}^{j}[1-\zeta(m)]\right\}. \tag{20}$$

One should picture the moving 1 as starting off with one unit of momentum, and then gaining an additional unit at each 1 encountered and losing a unit at each 0 until the particle has zero momentum, at which time it stops. The symmetric rule holds for jumps to sites $k<i$. It is clear that a singularity in $D(\rho)$ will occur when $\rho=\frac{1}{2}$; this is nonrigorous, but if the system possesses any element of ergodicity then for any density in excess of $\frac{1}{2}$ there is positive probability that a 1 will move an infinite distance.

The singularity at $\rho_c=\frac{1}{2}$ is in fact observed numerically, with a pole of order $\phi=2.25\pm0.15$. Simulations also show that the system is sustaining a phase transition in the sense of diverging spin-spin correlations at $\rho_c=\frac{1}{2}$. It is difficult to independently extract the value of ν directly from numerical measurements of the correlation function. The divergence appears to be rather slow, and does not give a good fit to a power law at the densities we consider. Instead, we use measurements of the fluctuations as a function of the density to extract the estimate $\nu(-2+\eta)=0.95\pm0.15$, which clearly indicates that fluctuations are suppressed at $\rho_c=\frac{1}{2}$ relative to the central-limit value. On the open version of the system (driven exactly as were the two-state models, above), substituting these exponents into (11) implies that fluctuations should dominate when $d_B\geq 0.19\pm0.22$.

Figure 5 shows the variation of the exponents a and b with driving exponent d_B. Here the density ρ_N and fluctuations s_N are measured in the middle half of the system. The solid lines are the predictions of Eqs. (6) and (9) for b and a, respectively, using the values $\phi=2.25$ and $\nu(-2+\eta)=0.95$. The predictions are consistent with the measured values in the regime where hydrodynamics is expected to hold.

Figure 6(a) illustrates a collapse of event-size distributions $P(n)$ as a function of density for the closed system. The data is rescaled by the distance from the singularity. Figure 6(b) illustrates a similar collapse for the open system at $d_B=-0.25$, which is well below the threshold at which fluctuations should destroy diffusion hydrodynamics. As for the BTW model, here we use the density in the middle half of the system to obtain the scaling col-

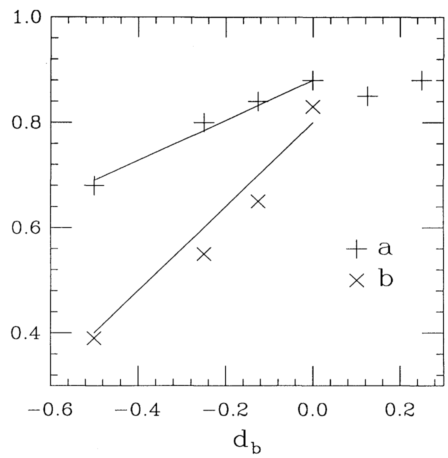

FIG. 5. Exponents a (describing the scaling of s_N with system size N) and b [scaling of $(\rho_c - \rho_N)$ with N] as a function of driving exponent d_B for the count model. The density and fluctuations have been measured in the middle half of an open system. The solid lines are the predictions of Eqs. (6) and (9) for b and a, respectively, where we have taken $\phi = 2.25$ and $\nu(-2+\eta) = 0.95$. These values fit the data reasonably well in the regime $d_B \lesssim 0$ in which hydrodynamics is expected to hold.

lapse. We again observe a mild discrepancy between the values of $\hat{\nu}$ which best collapse the data on the open and closed systems. The discrepancy decreases as the system size is increased, so that, as before, we attribute the difference to spatial variations of the density on the open system rather than a breakdown of local equilibrium. Figure 6(c) shows the results for the open system with $d_B = 0.25$, which is in the hard driving regime where hydrodynamics has broken down due to fluctuations. In this case the distribution has been rescaled by the system size. As in the BTW model when fluctuations are relevant, in this case the event-size distribution is essentially a pure power law with $\hat{\xi} \sim N$. For the count model we also observe an excess of systemwide events. We have observed a similar excess in the BTW model with $d_B = 2$ imposing periodic boundary conditions in x and open boundaries in y.

D. Limited local sandpile models

Finally, we consider the one-dimensional limited local model, introduced and studied in Ref. [2]. This model and the BTW sandpile model discussed above are the two most studied SOC models to date. The model is an N-site integer lattice, and each site i has an associated height $h(i)$ that represents the number of grains on the site. Each site also has a slope $z(i) \equiv h(i) - h(i+1)$. Slope is the relevant conserved quantity, specifically because one can study the equilibrium statistical mechanics for closed systems in terms of slope. The rules are as follows. Sand is added one grain at a time to a randomly chosen site. If the slope at a given site i is above a threshold slope $z_c = 2$, then two grains fall to the neighboring site: $h(i) \rightarrow h(i) - 2$ and $h(i+1) \rightarrow h(i) + 2$. The rules can be

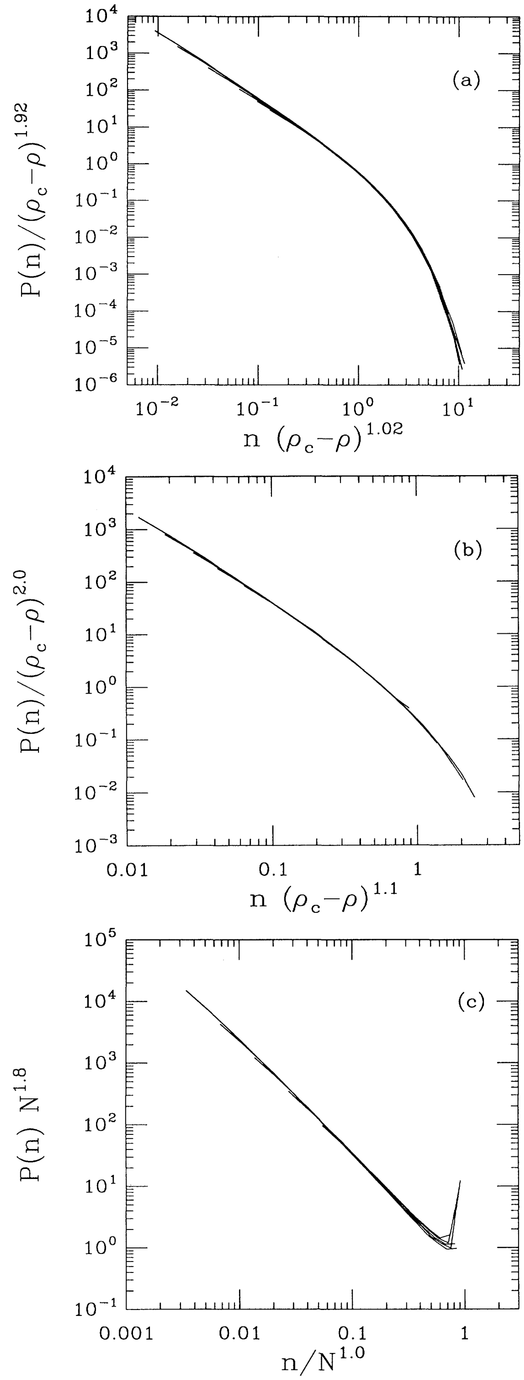

FIG. 6. Scaling collapse of the event distribution $P(n)$ for the count model. (a) shows the results for the closed system with $N = 8192$, and (b) is the open system with $N = 64,128,256,512,1024$ and $d_B = -0.25$, which is below the threshold at which fluctuations dominate. In each case the data is rescaled by the distance from the singularity. From (a) and (b) we estimate $\hat{\nu} = 1.0 \pm 0.1$. (c) is the open system with $N = 64,128,256,512,1024$ and $d_B = 0.25$, which is in the hard driving regime where hydrodynamics has broken down due to fluctuations. In this case we have collapsed our data with respect to system size N as in Fig. 4(c). The distribution is a pure power law out to the system size, at which point there are an excess of systemwide events.

written purely in terms of slope where the concept of "troughs" becomes evident [17].

Previously it was found numerically that $\phi=4$, a result that was later obtained analytically in Ref. [8]. For the usual open version of this system, addition of grains results in a net drift of slope $[z(i)=z(i)+1$ and $z(i-1)=z(i-1)-1$, so slope flows from left to right when sand is added at site $i]$. We remove this drift by symmetrizing the addition rule, which does not change the singularity. The system is then driven in the same way as the two-state model, injecting slope at the left boundary and allowing slope to leave the system at the open right boundary. This avoids pathologies associated with the usual boundary conditions and allows us to tune the driving rate to scale as N^{d_B}.

On the closed system it appears that fluctuations are suppressed, even away from ρ_c. Measurements of the density fluctuations in boxes up to size 512 on a system of size 8192 yields $s_N \sim N^{-a}$ with $a \approx 0.8$, well above the central limit value of $\frac{1}{2}$, independent of the distance from the singularity. This result is surprising: if there are no diverging length scales away from ρ_c then fluctuations must eventually be central limit as N is increased. The fact that we still observe suppressed fluctuations suggests that there is a very large (possibly finite) correlation length in the system which has not been reached at the system sizes considered here.

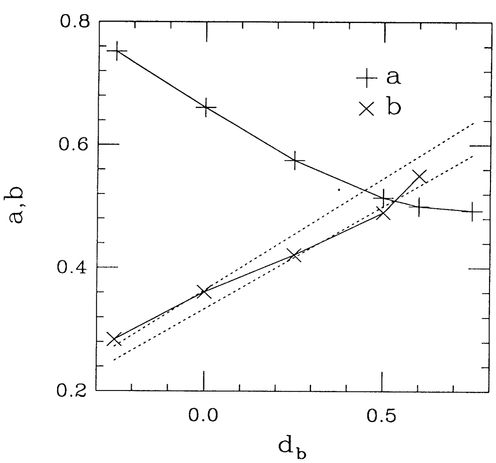

FIG. 7. Exponents a (describing the scaling of s_N with system size) and b [scaling of $(\rho_c-\rho_N)$ with system size] as a function of driving exponent d_B for the limited local model. The density and fluctuations have been measured in the middle half of an open system. The two dashed lines are the predictions of Eq. (6) when $d=1$: $b=(1+d_B)/(\phi-1)$. The lower dashed line is for $\phi=4$ (the value measured in the closed system), the upper dashed line is $\phi=3.75$. Note that neither value of ϕ fits the data for b, even in the regime where $b<a$. The breakdown of local equilibrium is clearly indicated by the values of a, which are clearly incompatible with Eq. (9), which would predict that when $a>\frac{1}{2}$, the slopes of a and b as a function of d_B should have the same sign. We note finally that the inequality $b \le a$ for fluctuations to become relevant still seems to have some content, since the smooth behavior of the curves changes above their crossing point at $d_B \approx 0.5$.

On the open system with the conventional driving, suppressed height fluctuations were observed earlier in Ref. [18]. We also observe suppressed fluctuations on the open driven system when the driving dimension $d_B=0$ corresponds to the driving dimensionality of the conventional case. However, in direct contradiction to the validity of local equilibrium, for the open system we find that as the system is driven harder (i.e., d_B is increased, so that convergence to the critical density is increased), the fluctuations become less suppressed, with a limiting central limit value of $\frac{1}{2}$ for large d_B. This is illustrated in Fig. 7, where we plot a and b as a function of d_B. If local equilibrium held, then a and b would both increase with increasing d_B. The decrease in a is a direct indication that this approximation has broken down. The fact that fluctuations actually increase suggests a structure for the stationary distribution in the open system that is very different from that of the closed system, even for very small d_B. We speculate that the long-range correlations observed on the closed system may ultimately provide an explanation of this phenomenon, as it may be that even a relatively small flux through the system will interfere with this large correlation length.

IV. CONCLUSIONS

Chronologically, this study began with a detailed numerical examination of the four sandpile models in Ref. [2]. In particular, we set out to measure the diffusion singularities and verify the consistency between the behavior on the closed and open systems. The experience was somewhat sobering, and it became clear that the behavior of these seemingly simple automata could be quite complex, involving not only a diffusion singularity but also a phase transition in the density. Furthermore, even that extra level of complexity does not sufficiently explain the range of behaviors observed in these models, as made evident by the lack of complete documentation for them in Table I. We have, however, numerically observed some very large length scales that may provide a partial answer, and we hope to have more to say on the subject in a later publication.

One of our first observations was that the unlimited models of Ref. [2], while exhibiting singular diffusion coefficients on the closed system, actually converge to the singularity from *above* on the open system, in apparent contradiction of the hydrodynamic description. It has since become clear that there are pathologies with the unlimited models on the open system—the systems get stuck in configurations with density above the singularity.

However, from this we learned that the hydrodynamic description could fail, and, in fact, should fail when fluctuations force the system over the singularity with non-negligible probability. We have presented (1) an explanation for why such fluctuations can invalidate the diffusion description, (2) a simple exponent inequality, involving standard critical exponents, which establishes when this occurs, (3) numerical evidence that shows that failures of the diffusion description is in fact observed under hard driving with palpable changes in avalanche distributions, and (4) numerical evidence for the failure of local equilibrium in the limited local model.

This leads to two important unresolved issues for future work. First, in the case where the diffusion description breaks down as a consequence of fluctuations which extend above the singularity, one may ask: What is the new description which replaces (1)? An answer to this question would, for example, assist in understanding the BTW model with the conventional driving ($d_B = 2$).

The second issue that remains unresolved is a quantitative understanding of the breakdown of local equilibrium. For example, it would be useful to develop an exponent inequality analogous to (11). The assumption of local equilibrium is relied upon heavily in the analysis of other nonequilibrium systems. Alternative descriptions have been developed for some fluids problems for conditions under which local equilibrium is not a good approximation [19]. It is not clear how to address this question for the SOC systems that we have considered here. Because of the diverging transition lengths, this failure may occur more readily in SOC systems, suggesting the possibility that they may provide a useful arena in which to examine the question of the breakdown of local equilibrium in driven systems.

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[15] While typically this assumption is valid for sandpile models, it does not apply to the models in Ref. [2], such as the limited local model, with the conventional driving. In that case, for slope both the open and closed boundary are the left edge, and there is a slope drift towards the right induced by the addition of sand. The modified version of the limited local model which we study in Sec. III overcomes this problem without altering the diffusion singularity.
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